FIRST EDITION

INTRODUCTION TO VIBRATION IN ENGINEERING

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By SUNG LEE AND NORMAN WERELEY

FIRST EDITION

INTRODUCTION TO VIBRATION IN ENGINEERING

BY SUNG W. LEE AND NORMAN M. WERELEY
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PREFACE

This book is written as a textbook for undergraduate and first-year graduate students who are interested primarily in aerospace engineering. We have had the privilege of teaching a course in aerospace vibration, including aeroelasticity, for a number of years. The challenge in developing a course of this type is that many of the existing textbooks focus on vibrations from a perspective different from that of an aerospace engineer.

The challenges faced by an aerospace engineer in studying vibrations are that vibrations are often caused by aerodynamic or motion-dependent forces. In addition, the aerospace system or structure undergoing vibration can be complex, with distributed mass and stiffness. For example, a wing or fuselage structure contains a stressed skin surrounding a multitude of spars and ribs that seem to defy simple analyses. Yet, the physical insights that are developed from a careful study of a single-degree-of-freedom system described by a simple, ordinary differential equation—that is, a mass and spring and possibly a damper—are profound, and in limited cases, can describe the bending vibration of such a wing. For systems with multi-degrees of freedom, the notion of a mode as a single-degree-of-freedom system described by an ordinary differential equation is manifested in the concept of the normal mode equations.

Therefore, a key objective of this book is to develop a solid foundation and understanding of single-degree-of-freedom systems that can be used as building blocks to tackle more difficult and general multi-degree-of-freedom systems. A second goal of this book is to introduce the Lagrange equation approach, which can be used to construct equations of motion for complex systems. Subsequently, simple models with two degrees of freedom are used to discuss the subject of dynamic instability, which can occur for such systems as a slender body under a follower force or an aircraft wing in flight. Interactions between the flexible structure and the follower force, including aerodynamic loads and inertia, can lead to dynamic instabilities. Another key goal of this book is to develop and illustrate the tools by which a structural body can be transformed via the finite element method into an equation of motion of finite degrees of freedom. This is done by using a slender body undergoing longitudinal, torsional, or bending motions as examples.

But perhaps the most important purpose of this book is to provide an approachable textbook for the student. For each topic covered, concepts are illustrated through examples and short explanations. Moreover, the goal is to show how to compute solutions, whether analytical or numerical, and to illustrate how the vibrations of complex aerospace structural vibrations can be reduced in complexity using a set of tools that are easily programmed in software such as MATLAB. Ultimately, the aerospace engineer seeks to develop analytical models, compute results, make assessments, and then decide what should be done to improve the performance of the aerospace structure. Therefore, the focus is on practical application of techniques, with the goal of computing solutions to enable the aerospace engineering design iteration.

The material in this book serves as the textbook for our senior-level one-semester class in Aerospace Vibrations. We have concurrently taught both a regular and an honors version of the class based on these materials each year. Some of the advanced topics can be easily omitted to slow the pace and pressure of the class.

As with any project of this kind, many colleagues and students have contributed to the overall content. We thank all of our students who have taken the class, our teaching assistants, and our graduate students and postdoctoral staff, all of whom have contributed to the various aspects of this book. In particular, we thank Dr. Soonwook Kwon, who provided tremendous help in editing the manuscript for this book.

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1 DYNAMIC RESPONSE OF SDOF SYSTEMS

Many physical systems undergoing dynamic motion can be modeled as a single-degree-of-freedom (SDOF) system with a mass, a spring, and a damper. A degree of freedom (DOF) is a linear displacement or an angular displacement that describes the motion of a system. Once a physical system is modeled as a collection of a mass, a spring, and a damper, one can construct an equation of motion via Newton's second law of motion or an energy approach. The equation motion appears as an ordinary differential equation (ODE) to which one can apply mathematical techniques to determine the system response.

In this chapter, we will construct an equation of motion for SDOF systems by applying Newton's second law of motion. We will then consider "free" vibration, in which a system undergoes vibrational motion with no (or free of) externally applied load. The free vibration starts with the "initial conditions" in terms of a displacement and a velocity specified at the start of the motion. We will find that free vibration occurs at a frequency and amplitude determined by the system parameters (such as mass, spring constant, and damping constant) and the initial conditions. The discussion on free vibration will then be followed by the study of forced vibration under externally applied loads such as step function load or impulse. Subsequently, we will introduce a numerical method that can be used for numerical integration of equation of motion in time to determine the response of SDOF systems subjected to any types of applied loads. The topics to be covered in this chapter are summarized as follows:

1.1 Equation of motion
1.2 Free vibration of undamped SDOF systems
1.3 Free vibration of damped SDOF systems
1.4 Forced vibration: step function load and impulse
1.5 Numerical methods to determine system response

In this chapter and the following chapters, we will adopt both the SI system of units and the English engineering system of units still used in the United States. The basic units in the SI system relevant to dynamics are kilogram (*kg*) for mass, meter (*m*) for length and second (sec or s) for time while the basic units in the English engineering system are *slug* for mass, foot (*ft*) for length and second (sec or s) for time. In the SI system a force of newton (*N*) is defined as $1\ N = 1\ kg \times 1\ m/\sec^2$. In the English engineering system a force of pound (*lb*) is defined as $1\ lb = 1\ slug \times 1\ ft/\sec^2$.

1.1 Equation of Motion

The equation of motion for a SDOF system with a mass, a spring, and a dashpot damper can be derived using Newton's second law.

Example 1.1.1 Spring-mass-damper system

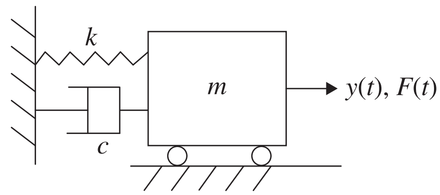


Figure 1.1 Mass-spring-dashpot damper system.

In Figure 1.1, concentrated mass m, spring constant k, and damping constant c represent the effective mass, stiffness, and energy-dissipating properties of the system, while y is the displacement or DOF or the system, and F is the applied load. Velocity and acceleration of the system are expressed as follows:

$$\dot{y} = \frac{dy}{dt}\text{: velocity,} \qquad \ddot{y} = \frac{d^2 y}{dt^2}\text{: acceleration}$$

To apply Newton's second law, we isolate the mass and draw a free-body diagram (FBD) as shown below.

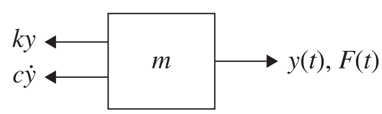


Figure 1.2 Free-body diagram.

In the FBD, ky is the restoring spring force, and $c\dot{y}$ is the restoring damping force. According to Newton's second law,

$$m\ddot{y} = F - ky - c\dot{y}$$

or

$$m\ddot{y} + c\dot{y} + ky = F \qquad (1.1.1)$$

Example 1.1.2 Effect of gravity

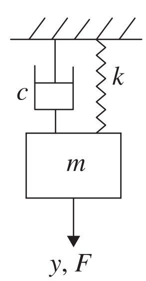


Figure 1.3 SDOF system under gravity effect.

Equation of motion is

$$m\ddot{y} + c\dot{y} + ky = mg + F \tag{1.1.2}$$

where g is the gravity constant and y is the displacement measured from the original length of the spring. For a mass at static equilibrium under gravity,

$$ky_s = mg \tag{1.1.3}$$

Displacement y can be expressed as

$$y = y_s + \hat{y} \tag{1.1.4}$$

where y_s: static displacement measured from the original length of the spring
$\hat{y}$: displacement measured from the static equilibrium position

Placing equation (1.1.4) into equation (1.1.2),

$$m(\ddot{y}_s + \ddot{\hat{y}}) + c(\dot{y}_s + \dot{\hat{y}}) + k(y_s + \hat{y}) = mg + F \tag{1.1.5}$$

Using equation (1.1.3) and noting that $\ddot{y}_s = \dot{y}_s = 0$

$$m\ddot{\hat{y}} + c\dot{\hat{y}} + k\hat{y} = F \tag{1.1.6}$$

Accordingly, when the displacement is measured from the static equilibrium position, the gravity term disappears from the equation of motion.

Example 1.1.3 Consider an instrument package of mass m installed in the nose cone of a rocket with a cushion against vibration. The cushion is represented by a spring k and a damper c. The rocket is fired vertically from rest with a given acceleration $\ddot{d}$. For this case, one can use a model of a SDOF system with a moving base as shown in Figure 1.4.

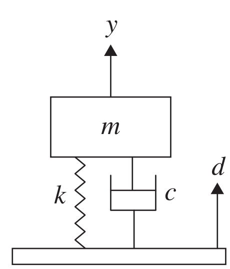


Figure 1.4 SDOF system with a moving base.

Isolating the mass (instrument) to draw a free-body diagram, taking into account the spring forces and damping forces acting on the mass, and applying Newton's second law,

$$m\ddot{y} = -ky + kd - c\dot{y} + c\dot{d} = -k(y - d) - c(\dot{y} - \dot{d}) \tag{1.1.7}$$

Introducing $y_R = y - d$,

$$m(\ddot{y}_R + \ddot{d}) = -ky_R - c\dot{y}_R \tag{1.1.8}$$

$$m\ddot{y}_R + c\dot{y}_R + ky_R = -m\ddot{d} \tag{1.1.9}$$

Equation (1.1.9) represents a SDOF system with displacement y_R under applied load $F = -m\ddot{d}$ in which the bottom end of the spring is fixed to the base.

1.2 Free Vibration of Undamped SDOF Systems

When a system vibrates with no external force (F = 0), the system is then said to undergo **free vibration**. For free vibration, the system starts the motion with given initial displacement and initial velocity, called **initial conditions**.

Let's consider free vibration of an undamped ($c = 0$) system.

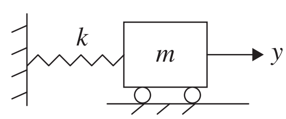


Figure 1.5 SDOF system with no damping.

Equation of motion: $m\ddot{y} + ky = 0$ (1.2.1)

Initial conditions:

initial displacement: $y(0) = y_0$

initial velocity: $\dot{y}(0) = \dot{y}_0$

Note:

1) y_0 and $\dot{y}_0$ represent given values.
2) The initial displacement corresponds to the initial strain energy stored in the spring $\left(= \frac{1}{2}ky_0^2\right)$.
3) The initial velocity corresponds to the initial kinetic energy of the mass $\left(= \frac{1}{2}m\dot{y}_0^2\right)$.

With no damping, total energy (= kinetic energy + strain energy) is conserved during free vibration. There is exchange of energy between the kinetic energy and the strain energy during vibration.

Equation (1.2.1) is a second-order (in time), homogeneous ordinary differential equation (ODE) with a solution of the following form:

$$y = Ae^{pt} \tag{1.2.2}$$

Placing equation (1.2.2) to equation (1.2.1),

$$(mp^2 + k)Ae^{pt} = 0$$

For nontrivial y (i.e., nonzero A),

$$mp^2 + k = 0 \tag{1.2.3}$$

Introducing

$$\omega_n = \sqrt{\frac{k}{m}} \tag{1.2.4}$$

Equation (1.2.3) can be expressed as

$$p^2 + \omega_n^2 = 0 \tag{1.2.5}$$

From equation (1.2.5),

$$p = \pm i\omega_n \;\rightarrow\; p_1 = i\omega_n \text{ and } p_2 = -i\omega_n \tag{1.2.6}$$

The general solution to equation (1.2.1) is then expressed as a linear combination of the two solutions corresponding to p_1 and p_2 in equation (1.2.6) as follows:

$$y = A_1 e^{p_1 t} + A_2 e^{p_2 t} = A_1 e^{i\omega_n t} + A_2 e^{-i\omega_n t} \tag{1.2.7}$$

Recall the following Euler's formula:

$$\begin{aligned} e^{iB} &= \cos B + i \sin B \\ e^{-iB} &= \cos B - i \sin B \end{aligned} \tag{1.2.8}$$

Using equation (1.2.8), equation (1.2.7) can be expressed as

$$y(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t \tag{1.2.9}$$

where C_1 and C_2 are a new set of constants. From equation (1.2.9),

$$\dot{y}(t) = -C_1 \omega_n \sin \omega_n t + C_2 \omega_n \cos \omega_n t \tag{1.2.10}$$

Constants C_1 and C_2 are determined by applying the initial conditions as follows:

At time $t = 0$, initial displacement: $y(0) = y_0$, and from equation (1.2.9),

$$C_1 = y_0 \tag{1.2.11}$$

Initial velocity is $\dot{y}(0) = \dot{y}_0$, and from equation (1.2.10),

$$C_2 \omega_n = \dot{y}_0 \rightarrow C_2 = \frac{\dot{y}_0}{\omega_n} \tag{1.2.12}$$

Alternately, equation (1.2.9) can be expressed as a single sinusoidal function in time, such that

$$y(t) = C \sin(\omega_n t + \phi) \tag{1.2.13}$$

where C: amplitude

ϕ : phase angle

ω_n : angle covered in one second, called *natural frequency* (rad/sec)

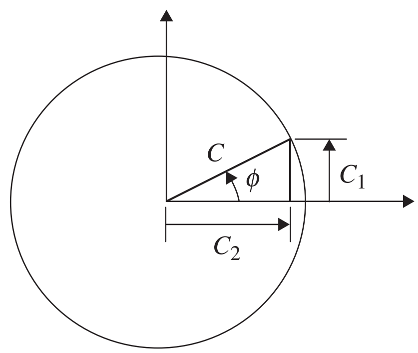
Amplitude and phase angle are related to the initial conditions as follows. Using one of the trigonometric formulas, equation (1.2.13) can be expressed as

$$y(t) = C(\sin\omega_n t\cos\phi + \cos\omega_n t\sin\phi) \tag{1.2.14}$$

Comparing equation (1.2.14) with equation (1.2.9),

$$\begin{aligned} C_1 &= C\sin\phi \\ C_2 &= C\cos\phi \end{aligned} \tag{1.2.15}$$

From equation (1.2.15) and also from the sketch,



$$\begin{aligned} C &= \sqrt{C_1^2 + C_2^2} = \sqrt{y_0^2 + \left(\frac{\dot{y}_0}{\omega_n}\right)^2} \\ \tan\phi &= \frac{C_1}{C_2} = \frac{y_0}{(\frac{\dot{y}_0}{\omega_n})} \qquad \rightarrow \qquad \phi = \tan^{-1}\frac{C_1}{C_2} \end{aligned} \tag{1.2.16}$$

<u>Note:</u>

1) An undamped SDOF system undergoes free vibration with natural frequency

$$\omega_n\left(=\sqrt{\frac{k}{m}}\right) \text{ rad/sec}$$

2) T: period and

$$\omega_n T = 2\pi \ \rightarrow \ T = \frac{2\pi}{\omega_n} \tag{1.2.17}$$

3) Natural frequency f in cycles or Hertz (Hz) is defined as follows:

$$f = \frac{1}{T} = \frac{\omega_n}{2\pi} \tag{1.2.18}$$

Example 1.2.1 Consider a spacecraft on the ground. For vertical motion, the spacecraft is modeled as a SDOF system with mass M and spring k. The spring represents the vertical stiffness of the landing gear. Under the deadweight of the spacecraft, the spring displaces vertically by 7.62 cm.

(a) Determine the natural frequency of free vibration in the vertical direction.
(b) Determine the natural frequency when the stiffness of the spring is doubled.

(a)

$$Mg = ky_{static} \rightarrow \frac{k}{M} = \frac{g}{y_{static}}$$

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{g}{y_{static}}} = \sqrt{\frac{9.81}{0.0762}} = 11.35 \text{ rad/sec}$$

$$f = \frac{\omega_n}{2\pi} = 1.806 \text{ Hz}$$

(b) For constant mass, $\omega_n \sim \sqrt{k}$ or $f \sim \sqrt{k}$

$$f = \sqrt{2} \times 1.806 = 2.554 \text{ Hz}$$

1.3 Free Vibration of Damped SDOF Systems

Consider a system that consists of a mass, a spring, and a damper.

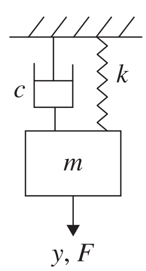


Figure 1.6 A damped SDOF system.

The dashpot with damping constant c represents an energy-dissipating mechanism. Newton's second law applied to the isolated mass:

$$\begin{aligned} m\ddot{y} &= F - ky - c\dot{y} \\ m\ddot{y} + c\dot{y} + ky &= F \\ \ddot{y} + \frac{c}{m}\dot{y} + \frac{k}{m}y &= \frac{F}{m} \end{aligned} \tag{1.3.1}$$

Let's now introduce a nondimensional damping ratio ζ as follows:

$$\frac{c}{m} = 2\varsigma\omega_n$$

Then,
$$\varsigma = \frac{c}{2m\omega_n} \tag{1.3.2}$$

The last of equation (1.3.1) can be rewritten as

$$\ddot{y} + 2\varsigma\omega_n\dot{y} + \omega_n^2 y = \frac{F}{m} \tag{1.3.3}$$

where

$$\omega_n^2 = \frac{k}{m}$$

For free vibration ($F = 0$),

equation of motion:
$$\ddot{y} + 2\varsigma\omega_n\dot{y} + \omega_n^2 y = 0 \tag{1.3.4}$$

initial conditions:
$$y(0) = y_0 \text{ and } \dot{y}(0) = \dot{y}_o \tag{1.3.5}$$

<u>Note:</u>

In the presence of damping, free vibration will die out eventually.

A solution to equation (1.3.4), which is homogeneous, is of the following form:

$$y(t) = Ae^{pt} \tag{1.3.6}$$

Then,

$$\dot{y}(t) = pAe^{pt}, \quad \ddot{y}(t) = p^2Ae^{pt} \tag{1.3.7}$$

Placing equations (1.3.6) and (1.3.7) into equation (1.3.4),

$$(p^2 + 2\varsigma\omega_n p + \omega_n^2)Ae^{pt} = 0 \tag{1.3.8}$$

For nontrivial y (i.e., nonzero A),

$$p^2 + 2\varsigma\omega_n p + \omega_n^2 = 0 \tag{1.3.9}$$

Then, from equation (1.3.9),

$$\begin{aligned}
&(p + \varsigma\omega_n)^2 - (\varsigma\omega_n)^2 + \omega_n^2 = 0 \\
&(p + \varsigma\omega_n)^2 = \omega_n^2(\varsigma^2 - 1) \\
&p + \varsigma\omega_n = \pm\omega_n\sqrt{\varsigma^2 - 1} \\
&p = -\varsigma\omega_n \pm \omega_n\sqrt{\varsigma^2 - 1}
\end{aligned} \tag{1.3.10}$$

Depending on the magnitude of the damping ratio, one can consider the following three cases:

1) Overdamped case ($\varsigma > 1$)

$$\begin{aligned} p_1 &= -\varsigma\omega_n + \omega_n\sqrt{\varsigma^2 - 1} \\ p_2 &= -\varsigma\omega_n - \omega_n\sqrt{\varsigma^2 - 1} \end{aligned} \tag{1.3.11}$$

For $\varsigma > 1$, both p_1 and p_2 are negative real numbers. The solution to the equation of motion in (1.3.4) is expressed as follows:

$$y(t) = c_1 e^{p_1 t} + c_2 e^{p_2 t} \tag{1.3.12}$$

Constants c_1 and c_2 are determined from the initial conditions. Note that, as time elapses, y drops to zero with no oscillation.

2) Critically damped case ($\varsigma = 1$)
For $\varsigma = 1$,

$$p_1 = p_2 = -\omega_n \tag{1.3.13}$$

The solution to the equation of motion in (1.3.4) is expressed as follows:

$$y(t) = c_1 e^{-\omega_n t} + c_2 t e^{-\omega_n t} \tag{1.3.14}$$

Constants c_1 and c_2 are determined from the initial conditions. Note that, as time elapses, y drops to zero with no oscillation.

3) Underdamped case ($\varsigma < 1$)
For $\zeta<1$,

$$\begin{aligned} p_1 &= -\varsigma\omega_n + i\omega_n\sqrt{1-\varsigma^2} = -\varsigma\omega_n + i\omega_d \\ p_2 &= -\varsigma\omega_n - i\omega_n\sqrt{1-\varsigma^2} = -\varsigma\omega_n - i\omega_d \end{aligned} \tag{1.3.15}$$

where

$$\omega_d = \omega_n\sqrt{1-\varsigma^2} \tag{1.3.16}$$

The solution to equation (1.3.4) is then expressed as follows:

$$\begin{aligned} y(t) &= A_1 e^{p_1 t} + A_2 e^{p_2 t} = A_1 e^{(-\varsigma\omega_n + i\omega_d)t} + A_2 e^{(-\varsigma\omega_n - i\omega_d)t} \\ &= e^{-\varsigma\omega_n t}\left(A_1 e^{i\omega_d t} + A_2 e^{-i\omega_d t}\right) \end{aligned} \tag{1.3.17}$$

or

$$y(t) = e^{-\varsigma\omega_n t}(C_1 \cos\omega_d t + C_2 \sin\omega_d t) \tag{1.3.18}$$

From equation (1.3.18) for $y(t)$,

$$\dot{y}(t) = -\varsigma\omega_n e^{-\varsigma\omega_n t}(C_1 \cos\omega_d t + C_2 \sin\omega_d t) + e^{-\varsigma\omega_n t}(-C_1\omega_d \sin\omega_d t + C_2\omega_d \cos\omega_d t) \tag{1.3.19}$$

Applying the initial conditions, i.e., equation (1.3.5) to equations (1.3.18) and (1.3.19),

$$\begin{aligned} & y(0) = C_1 = y_0 \\ & \dot{y}(0) = -\varsigma\omega_n C_1 + C_2\omega_d = \dot{y}_0 \\ & \rightarrow C_2 = \frac{\dot{y}_0 + \varsigma\omega_n y_0}{\omega_d} \end{aligned} \tag{1.3.20}$$

Response $y(t)$ in equation (1.3.18) can be rewritten into a single sinusoidal function of time, such that

$$y(t) = e^{-\varsigma\omega_n t} C \sin(\omega_d t + \phi) \tag{1.3.21}$$

where $e^{-\varsigma\omega_n t}C$: amplitude that decays exponentially

ϕ : phase angle

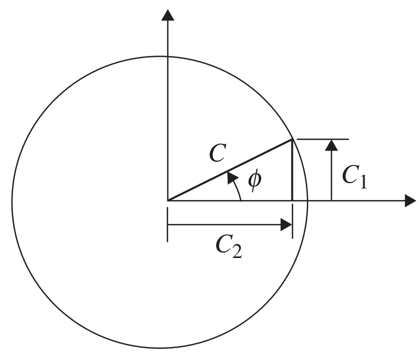
Using one of the trigonometric formulas, equation (1.3.21) can be expressed as

$$y(t) = e^{-\varsigma\omega_n t} C(\sin\omega_d t \cos\phi + \cos\omega_d t \sin\phi) \tag{1.3.22}$$

Comparing equation (1.3.22) with equation (1.3.18),

$$C_1 = C\sin\phi, \quad C_2 = C\cos\phi \tag{1.3.23}$$

From equation (1.3.23) and also from the sketch,



$$\begin{aligned} C &= \sqrt{C_1^2 + C_2^2} \\ \tan\phi &= \frac{C_1}{C_2} \;\rightarrow\; \phi = \tan^{-1}\frac{C_1}{C_2} \end{aligned} \tag{1.3.24}$$

Recall:

$$C_1 = y_0, \quad C_2 = \frac{\dot{y}_0 + \varsigma\omega_n y_0}{\omega_d} \tag{1.3.25}$$

Note:

1) As time elapses, y approaches zero, i.e., free vibration dies out.
2) For ζ = 0, the above solution reduces to that for free vibration of an undamped system.
3) $\omega_d = \omega_n\sqrt{1-\varsigma^2}$: frequency of damped free vibration

   When ς^2 is very small compared with 1, $\omega_d \approx \omega_n$.
   For example, for ζ = 0.1, $\omega_d = 0.995\omega_n$.
4) $\omega_d T = 2\pi$ where T is the period.

Equation (1.3.21) can be expressed as

$$y(t) = Y_0 \sin(\omega_d t + \phi) \tag{1.3.26}$$

where $Y_0 = e^{-\varsigma\omega_n t}C$ is the amplitude that decays exponentially in time.

Then,

$$\ln Y_0 = -\varsigma\omega_n t + \ln C \tag{1.3.27}$$

At two different times, t_1 and t_2, equation (1.3.27) can be expressed as

$$\begin{aligned} \ln Y_1 &= -\varsigma\omega_n t_1 + \ln C \\ \ln Y_2 &= -\varsigma\omega_n t_2 + \ln C \end{aligned} \tag{1.3.28}$$

where

$$Y_1 = (Y_0)_{t=t_1}, \quad Y_2 = (Y_0)_{t=t_2} \tag{1.3.29}$$

From equation (1.3.28),

$$\ln Y_1 - \ln Y_2 = \varsigma\omega_n(t_2 - t_1) \tag{1.3.30}$$

$$\rightarrow \ln\frac{Y_1}{Y_2} = \varsigma\omega_n(t_2 - t_1) \tag{1.3.31}$$

or

$$\ln\frac{Y_1}{Y_2} = \frac{\varsigma}{\sqrt{1-\varsigma^2}}\omega_d(t_2 - t_1) \tag{1.3.32}$$

From equation (1.3.32),

$$\zeta = \frac{\delta}{\sqrt{(2\pi n)^2 + \delta^2}} \tag{1.3.33}$$

where

$$\delta = \ln\frac{Y_1}{Y_2} \tag{1.3.34}$$

and n is the number of cycles between t_1 and t_2 (i.e. $t_2 - t_1 = nT$). For $n = 1$, δ is called "logarithmic decrement". Equation (1.3.33) can be used to determine damping ratio ζ as shown in the following example.

Example 1.3.1 Consider an assembly of a sting balance with a model of a flight vehicle placed on a vibration test rig. One end of the sting balance is attached to the rig, while the other end is free. The vibration of the assembly in the vertical direction is modeled as a SDOF system.

In order to determine effective static and dynamic properties of the model, initially a static force is applied vertically at the free tip of the assembly by hanging a weight. It turns out that the static tip displacement of the assembly is 0.2 in. for a weight of 60 lbs. The cable connecting the weight to the assembly is then cut to initiate a free vibration. The records of ensuing free vibration of the tip show that, after three cycles, the elapsed time is 1.5 sec and the amplitude is 0.1 in. Determine the following for the sting balance.

- **(a)** Effective spring constant k (lb/ft).
- **(b)** Effective damping ratio ς.
- **(c)** Undamped natural frequency (rad/sec and Hz).
- **(d)** Effective mass m (slugs).
- **(e)** Effective damping coefficient c (lb-sec/ft).
- **(f)** Energy loss (in percentage) of the system at the end of three cycles.

(a) $k = \dfrac{60\ lb}{0.2/12\ ft} = 3{,}600\ \text{lb/ft}$

(b) $\delta = \ln\dfrac{Y_1}{Y_2} = \ln\dfrac{0.2}{0.1} = \ln 2 \quad \text{and} \quad n = 3,$

$$\zeta = \frac{\delta}{\sqrt{(2\pi n)^2 + \delta^2}} = \frac{\delta}{\sqrt{(6\pi)^2 + \delta^2}} = \quad 0.0367$$

(c) $T = \dfrac{1.5}{3} = 0.5\text{sec}, \quad \omega_d = \dfrac{2\pi}{T} = 12.566\ \text{rad/sec}$

$$\omega_n = \frac{\omega_d}{\sqrt{1-\varsigma^2}} = 12.567\ \text{rad/sec}$$

(d) $\omega_n = \sqrt{\dfrac{k}{m}} \ \rightarrow\ m = \dfrac{k}{\omega_n^2} = \dfrac{3{,}600}{(12.567)^2} = 22.795\ \text{slug}$

(e) $c = m(2\varsigma\omega_n) = 21.03\ \text{lb-sec/ft}$

(f) $\dfrac{\Delta E}{E_1} = \dfrac{E_1 - E_2}{E_1} = 1 - \left(\dfrac{Y_2}{Y_1}\right)^2 = 1 - \left(\dfrac{0.1}{0.2}\right)^2 = 0.75 \rightarrow 75\%$

1.4 Forced Vibration

Now, consider SDOF systems that are subjected to external load.

Equation of motion:

$$m\ddot{y} + c\dot{y} + ky = F \tag{1.4.1}$$

We can express $y(t)$ as follows:

$$y = y_H + y_p \tag{1.4.2}$$

where y_H is the solution to the following homogeneous equation:

$$m\ddot{y} + c\dot{y} + ky = 0 \tag{1.4.3}$$

and y_p is the particular solution to equation (1.4.1).

Then, for $\zeta < 1$,

$$y(t) = e^{-\varsigma\omega_n t}(C_1 \cos\omega_d t + C_2 \sin\omega_d t) + y_p \tag{1.4.4}$$

where C_1 and C_2 are determined from the initial conditions.

1.4.1 A SDOF System Subjected to a Step Load

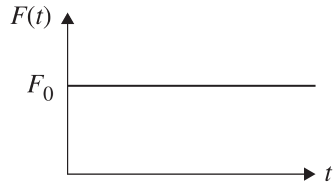


Figure 1.7 A step load of $F(t) = F_0$ constant in time.

The system is initially at rest. Assuming no damping, the equation of motion is

$$m\ddot{y} + ky = F_0 \tag{1.4.5}$$

and the initial conditions at $t = 0$ are

$$\begin{aligned} y(0) &= 0 \\ \dot{y}(0) &= 0 \end{aligned} \tag{1.4.6}$$

The homogeneous solution and a particular solution are as follows:

$$\begin{aligned} y_H &= C_1 \cos\omega_n t + C_2 \sin\omega_n t \\ y_P &= \frac{F_0}{k} \end{aligned} \tag{1.4.7}$$

Accordingly,

$$y = C_1 \cos\omega_n t + C_2 \sin\omega_n t + \frac{F_0}{k} \tag{1.4.8}$$

and

$$\dot{y} = \omega_n(-C_1 \sin\omega_n t + C_2 \cos\omega_n t) \tag{1.4.9}$$

Applying initial conditions to equations (1.4.8) and (1.4.9),

$$\text{equation (1.4.8)}:\ C_1 + \frac{F_0}{k} = 0 \ \rightarrow\ C_1 = -\frac{F_0}{k} \tag{1.4.10}$$

$$\text{equation (1.4.9)}:\ C_2 = 0 \tag{1.4.11}$$

Placing equations (1.4.10) and (1.4.11) into equation (1.4.8),

$$y = \frac{F_0}{k}(1 - \cos\omega_n t) \tag{1.4.12}$$

Now consider the case in which the same SDOF system is subjected to a static load of F_0. Then, the response is static, and

$$\begin{aligned} &ky_{STATIC} = F_0 \\ &\rightarrow y_{STATIC} = \frac{F_0}{k} \end{aligned} \tag{1.4.13}$$

Equation (1.4.12) can then be expressed as

$$y = y_{STATIC}(1 - \cos \omega_n t) \qquad (1.4.14)$$

From equation (1.4.14), one can see that under the dynamic condition

$$\max \ |y|_{DYNAMIC} = 2y_{STATIC} \qquad (1.4.15)$$

Also, under dynamic condition,

$$\max.(\text{spring force})_{DYNAMIC} = k(\max \ |y|_{DYNAMIC}) = k(2y_{STATIC}) = 2F_0 \qquad (1.4.16)$$

So,

$$\text{dynamic overload} = \ 2F_0 - F_0 = F_0 \qquad (1.4.17)$$

This simple example illustrates the importance of dynamic effect.

Example 1.4.1 Consider a spacecraft rotating to a specified angular orientation using an applied control moment. The spacecraft is initially at rest. For a rigid spacecraft under applied moment T,

$$I\ddot{\theta} = T$$

where I is the moment of inertia of the spacecraft. The control moment is expressed as

$$T = -K(\theta - \theta_F) - C\dot{\theta}$$

where θ_F is the desired final angle, and K and C are control gains. Then, the equation of motion is

$$I\ddot{\theta} + C\dot{\theta} + K\theta = K\theta_F$$

The solution to the above equation of motion can be expressed as

$$\theta(t) = e^{-\varsigma\omega_n t}(\mathrm{C}_1 \cos \omega_d t + \mathrm{C}_2 \sin \omega_d t) + \theta_F$$

Applying the initial conditions, one can find that

$$\mathrm{C}_1 = -\theta_F, \quad \mathrm{C}_2 = -\left[\frac{\varsigma}{\sqrt{1-\varsigma^2}}\right]\theta_F$$

Alternatively,

$$\theta = e^{-\varsigma\omega_n t}\hat{\mathrm{C}} \sin(\omega_d t + \phi) + \theta_F$$

where

$$\hat{\mathrm{C}} = \sqrt{\mathrm{C}_1^2 + \mathrm{C}_2^2}, \quad \phi = \tan^{-1}\frac{\mathrm{C}_1}{\mathrm{C}_2}$$

1.4.2 A Step Load of Finite Duration

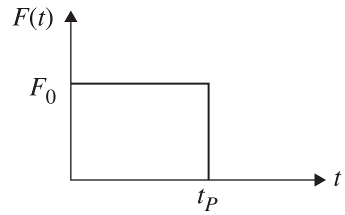


Figure 1.8 A step load of finite duration.

$$
\begin{aligned}
F(t) &= F_0 \qquad 0 \le t \le t_p \\
&= 0 \qquad t > t_p
\end{aligned}
\tag{1.4.18}
$$

For $0 \le t \le t_P$, equation (1.4.12) of Section 1.4.1 still holds.
Applied force disappears when $t > t_P$. Accordingly, for $t > t_P$, the system undergoes free vibration with the "initial" conditions at $t = t_P$.

From equation (1.4.12) of Section 1.4.1, at $t = t_P$,

$$
\begin{aligned}
y(t_P) &= \frac{F_0}{k}(1 - \cos\omega_n t_P) \\
\dot{y}(t_P) &= \frac{F_0}{k}\omega_n \sin\omega_n t_P
\end{aligned}
\tag{1.4.19}
$$

For undamped free vibration, the equation of motion is

$$m\ddot{y} + ky = 0 \tag{1.4.20}$$

Recall that the solution to the above homogeneous equation is

$$y = C_1 \cos\omega_n t + C_2 \sin\omega_n t \tag{1.4.21}$$

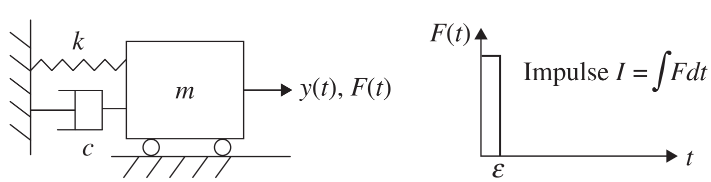
and

$$\dot{y} = \omega_n(-C_1 \sin\omega_n t + C_2 \cos\omega_n t) \tag{1.4.22}$$

Applying the initial conditions (for free vibration) at $t = t_P$ given in equation (1.4.19), one can show that

$$
\begin{aligned}
C_1 &= \frac{F_0}{k}(\cos\omega_n t_P - 1) \\
C_2 &= \frac{F_0}{k}\sin\omega_n t_P
\end{aligned}
\tag{1.4.23}
$$

1.4.3 Impulse Loading



The system is initially at rest, and the force is applied over $0 \le t \le \varepsilon$. Let's consider the case in which $\varepsilon \to 0$ while I is held constant.

Equation of motion:
$$m\ddot{y} + c\dot{y} + ky = F \tag{1.4.24}$$

Integrating the above equation from $t = 0$ to $t = \varepsilon$,

$$m\int_{t=0}^{t=\varepsilon} \ddot{y}\,dt + c\int_{t=0}^{t=\varepsilon} \dot{y}dt + k\int_{t=0}^{t=\varepsilon} ydt = \int_{t=0}^{t=\varepsilon} Fdt \tag{1.4.25}$$

For the right-hand side of equation (1.4.25),

$$\int_{t=0}^{t=\varepsilon} Fdt = I \tag{1.4.26}$$

For the integrals on the left,

$$k\int_{t=0}^{t=\varepsilon} ydt = 0 \tag{1.4.27}$$

$$c\int_{t=0}^{t=\varepsilon} \dot{y}dt = c[y(\varepsilon) - y(0)] = c[0 - 0] = 0 \tag{1.4.28}$$

because the mass has no time to move. For the first integral

$$m\int_{t=0}^{t=\varepsilon} \ddot{y}\,dt = m[\dot{y}(\varepsilon) - \dot{y}(0)] = m[\dot{y}(\varepsilon) - 0] = m\dot{y}(\varepsilon) \tag{1.4.29}$$

Then, equation (1.4.25) becomes

$$m\dot{y}(\varepsilon) = I \tag{1.4.30}$$

or
$$\dot{y}(\varepsilon) = \frac{I}{m} \tag{1.4.31}$$

So, the mass has not moved, but it has acquired initial velocity at $t = \varepsilon$. Accordingly, noting that $\varepsilon \to 0$, the system response after the impulse can be treated as a vibration problem with a "new" set of initial conditions as

$$y(0) = 0,\ \dot{y}(0) = \frac{I}{m} \tag{1.4.32}$$

Note that we can come to the same conclusion using equation (1.4.19) as follows: Let time t_p approach zero while keeping $I = F_0 t_p$ constant.

Example 1.4.2 A small spacecraft is dropped from a tower for a landing test. The vertical motion of the spacecraft after landing impact is modeled as a SDOF system subject to a given impulse I. Assume that the spacecraft does not topple after landing. The equation of motion is

$$m\ddot{y} + c\dot{y} + ky = mg$$

where m is the spacecraft mass, and c and k represent the damping constant and the spring constant of the landing gear. The vertical displacement y is defined positive downward. The solution to the above equation is

$$y(t) = e^{-\varsigma\omega_n t}(C_1 \cos\omega_d t + C_2 \sin\omega_d t) + \frac{mg}{k}$$

Then,

$$\dot{y}(t) = -\varsigma\omega_n e^{-\varsigma\omega_n t}(C_1 \cos\omega_d t + C_2 \sin\omega_d t) + e^{-\varsigma\omega_n t}(-C_1\omega_d \sin\omega_d t + C_2\omega_d \cos\omega_d t)$$

At touchdown $(t = 0)$, one can apply the initial conditions given as in equation (1.4.32),

$$0 = C_1 + \frac{mg}{k}, \ \frac{I}{m} = -\varsigma\omega_n C_1 + C_2\omega_d$$

from which one can obtain

$$C_1 = -\frac{mg}{k}, \ C_2 = \frac{1}{\omega_d}\left(\frac{I}{m} - \varsigma\omega_n \frac{mg}{k}\right)$$

The static equilibrium position after the vibration dies out is

$$y_{static} = \frac{mg}{k}$$

1.5 Numerical Methods to Determine System Response

Consider a SDOF system represented by

equation of motion: $$m\ddot{y} + f(\dot{y}, y) = F \quad (1.5.1)$$

initial conditions: $$y(0) = y_0, \ \dot{y}(0) = \dot{y}_0 \quad (1.5.2)$$

Note that

$$f(\dot{y}, y) = c\dot{y} + ky \quad (1.5.3)$$

for the damping force and the spring force proportional to $\dot{y}$ and y respectively. The system response under arbitrary load $F(t)$ can be determined by numerical methods. For this, one may transform the above equation into a system of two first-order equations by introducing a new variable v, defined as follows:

$$\dot{y} = v \quad (1.5.4)$$

Then, equation (1.5.1) can be expressed as

$$m\dot{v} + f(v, y) = F \quad (1.5.5)$$

or

$$\dot{v} = \frac{1}{m}\left(F - f(v, y)\right) \quad (1.5.6)$$

Equations (1.5.4) and (1.5.6) can be combined in to a single matrix equation as follows:

$$\begin{Bmatrix} \dot{y} \\ \dot{v} \end{Bmatrix} = \begin{Bmatrix} v \\ \dfrac{1}{m}\left(F - f(v,y)\right) \end{Bmatrix} \tag{1.5.7}$$

or

$$\dot{\mathbf{x}} = \hat{\mathbf{f}}(\mathbf{x}, F) \tag{1.5.8}$$

where

$$\mathbf{x} = \begin{Bmatrix} y \\ v \end{Bmatrix}, \quad \dot{\mathbf{x}} = \begin{Bmatrix} \dot{y} \\ \dot{v} \end{Bmatrix} \tag{1.5.9}$$

$$\hat{\mathbf{f}}(\mathbf{x}, F) = \begin{Bmatrix} v \\ \dfrac{1}{m}\left(F - f(v,y)\right) \end{Bmatrix} \tag{1.5.10}$$

Note that equation (1.5.7) or (1.5.8) is a first-order equation. The "initial" condition for $\mathbf{x}$ is as follows:

$$\mathbf{x}(0) = \begin{Bmatrix} y(0) \\ v(0) = \dot{y}(0) \end{Bmatrix} = \begin{Bmatrix} y_0 \\ \dot{y}_0 \end{Bmatrix} \tag{1.5.11}$$

There are many numerical schemes that can be used to determine $\mathbf{x}$ as a function of time. We will consider now the simplest scheme called the **Euler method**.

Notations:

$\mathbf{x}_n = \mathbf{x}(t_n)$: $\mathbf{x}$ at time t_n, $\quad \dot{\mathbf{x}}_n = \dot{\mathbf{x}}(t_n)$: $\dot{\mathbf{x}}$ at time t_n,
$t_{n+1} = t_n + \Delta t$, Δt: time increment
$\mathbf{x}_{n+1} = \mathbf{x}(t_{n+1})$: $\mathbf{x}$ at time t_{n+1}

For the Taylor expansion around time t_n,

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \dot{\mathbf{x}}_n \Delta t + \frac{1}{2}\ddot{\mathbf{x}}_n (\Delta t)^2 + \cdots\cdots \tag{1.5.12}$$

The Euler method assumes that

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \dot{\mathbf{x}}_n \Delta t \tag{1.5.13}$$

where

$$\dot{\mathbf{x}}_n = \hat{\mathbf{f}}(\mathbf{x}_n, F_n) = \begin{Bmatrix} v_n \\ \dfrac{1}{m_n}\left(F_n - f(v_n, y_n)\right) \end{Bmatrix} \tag{1.5.14}$$

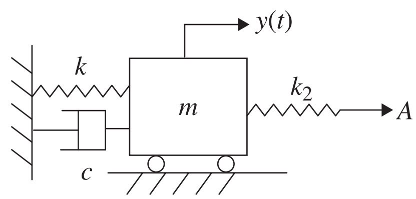
Given Δt, one can use equation (1.5.13) with equation (1.5.14) to determine $\mathbf{x}_{n+1}$. The entire process works as follows:

Choose a small Δt.
Set $n = 0$ and use equation (1.5.13) to determine $\mathbf{x}_1$ at $t_1 = \Delta t$.
Set $n = 1$ and use equation (1.5.13) to determine $\mathbf{x}_2$ at $t_2 = t_1 + \Delta t$.
Set $n = 2$ and use equation (1.5.13) to determine $\mathbf{x}_3$ at $t_3 = t_2 + \Delta t$.
Continue marching in time by increasing n until you obtain the solutions over the time span of your interest.

Note:
Higher order methods such as the fourth-order Runge-Kutta (RK4) method provide improved accuracy.

Chapter 1 Problem Sets

1.1 For the SDOF system shown below, draw a free-body diagram and set up the equation of motion for y. Identify the applied force term in the equation of motion. Note that A is a given displacement.



1.2 Consider a spacecraft undergoing vibration test on the ground. For vertical vibration, the spacecraft is modeled as a SDOF system. The landing gear consists of gas balloons. The natural frequency of the free vibration is measured at a given pressure. Subsequently, the gas pressure is reduced, and the natural frequency is measured. It turns out that the new frequency is reduced to 80% of the original frequency.

(a) Estimate the reduction in the stiffness of the system by determining a nondimensional quantity R defined as follows:

$$R = \frac{k_{old} - k_{new}}{k_{old}}$$

(b) Estimate the increase (in percent) in the static displacement.

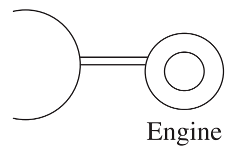
1.3 The wingtip bending vibration of an aircraft is modeled as a single DOF system with a spring and a mass. The wing is fixed at the root. The natural frequency of the wing is measured to be 6.1 Hz. When a fuel tank weighing 5,400 N is added at the wingtip, the natural frequency reduces to 5.2 Hz. Neglect damping effect.

(a) Determine the effective mass m_1 (not including the fuel tank) of the model.
(b) Determine the effective stiffness k of the model.

$y(t)$: vertical displacement of wingtip
m_1: effective mass of the wing, m_F: fuel tank mass
$m_2 = m_1 + m_F$

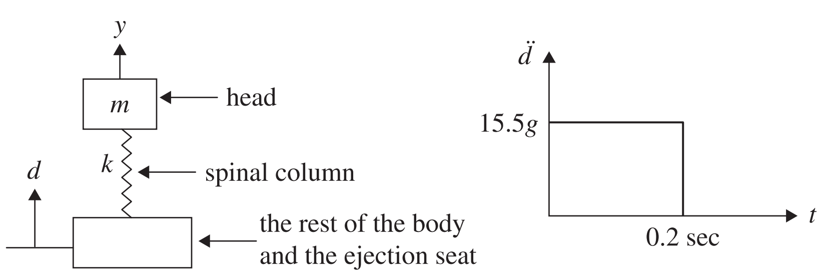
(c) Determine the natural frequency (in rad/sec and Hz) when the fuel tank weight is reduced to 3,500 N. due to fuel consumption.

1.4 Consider free vibration test of an aircraft engine mounted to the rear fuselage. The engine weighs 1,300 lb. The measured frequency of free vibration is 50 Hz, and the displacement amplitude of free vibration reduces to 50% of the initial value in four cycles. Ignore the weight of the engine mount and do the following:



(a) Estimate the effective damping ratio ζ.
(b) Estimate the effective stiffness k (lb/ft) of the engine mount.
(c) Suppose the engine is replaced with a new engine weighing 1,000 lb. Estimate the new measured frequency of free vibration. Also, determine the new damping ratio and estimate the reduction in the displacement amplitude of free vibration after 4 cycles.

1.5 Consider a SDOF model for dynamics of pilot ejection. The head is modeled as a single mass of m. The spinal column supporting the head is modeled as a spring with k. Damping is neglected. The time history of the ejection seat acceleration $\ddot{d}$ is approximated as shown in the figure.

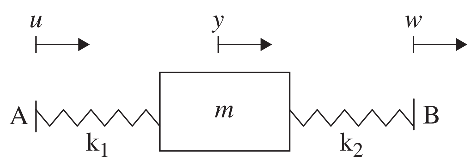


(a) Write the equation of motion in terms of the relative displacement $y_R = y - d$.
(b) Determine y_R of the head for $t < 0.2$ sec.
(c) Determine the absolute acceleration $\ddot{y}$ of the head. What is the peak value?
(d) Determine the force on the spinal column. What is the peak value?

1.6 A small UAV on takeoff run may hit a bump, and ensuing free vibration in the vertical direction can be modeled using a SDOF system of a mass, a spring, and a damper. The spring and the damper represent the landing gear. It is desired that the period of damped free vibration is 2.5 sec. and the amplitude of vertical displacement reduces to one-fourth in one cycle. The mass of the UAV is 1,300 kg. Determine the following for the landing gear.

(a) damping ratio ζ, **(b)** damping constant c, **(c)** spring constant k.

1.7 A single block of mass, 100 kg is attached to movable points A and B through springs as shown below. The displacements of points A and B are controlled externally and are denoted by u and w, respectively. Mass, m, itself has a displacement y. k_1 = 20000 N/m, k_2 = 30000 N/m

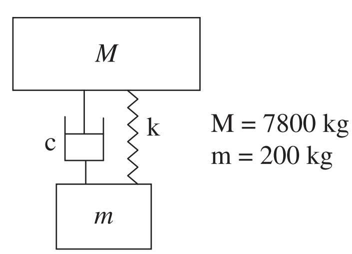


a) Draw the free-body diagram and determine the dynamic equation of motion of the mass. How must the points A and B move in order for the mass to remain in static equilibrium at all times? State the condition in terms of u and w.

b) If the points A and B are fixed, i.e., $u = w = 0$, what is the natural frequency of vibration of the mass, m?

c) Points A and B are fixed, and the mass is hit with an impulse, I, at time t = 0. If both springs have a load limit of 1000 N, what is the maximum value of I for the springs to remain intact? If I is slightly increased above this value, which spring will break first?

1.8 A helicopter has a gross mass of 8000 kg. The main landing gear with a mass of 200 kg is connected to the mid-fuselage by a spring and a damper. A simple model is sketched below.



Prior to takeoff, the landing gear is in contact with the ground, and the static deflection of the spring is 25 cm.

a) What is the value of the spring constant, k?

b) What is the natural frequency of the helicopter on its landing gear in units of Hz? Neglect damping.

During steady flight, the helicopter is flying at a constant altitude.

c) What is the extension of the spring under static equilibrium?

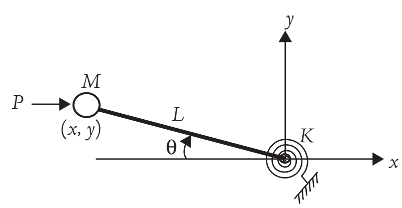
d) What is the natural frequency, in Hz, of the landing gear attached to the helicopter during steady flight? Neglect damping.

During landing, the helicopter is descending at a constant rate of 1m/s. Assume that the tires remain in contact with the ground after touchdown. Let $t = 0$ at the point of touchdown.

- **e)** Draw the free-body diagram of the helicopter, and obtain the dynamic equation of motion of the helicopter in terms of parameters M, k, and c. (Hint: This is a forced vibration problem with a constant external force due to gravity.)
- **f)** What are the initial conditions for the helicopter? (Note: Be wary of signs.) Compute the displacement and the velocity of the helicopter as a function of time, t. Neglect damping.
- **g)** Repeat (f) for the value of damping coefficient, $\zeta = 0.2$.
- **h)** Plot the trajectory of the helicopter on a graph of displacement versus velocity for both (h) and (i). What do you observe? What can you say about the total energy of the system in both cases?

1.9 Consider a system of a concentrated mass M, a torsional spring with constant K and a rigid bar as shown in the sketch. The sketch shows an initially horizontal bar of length L in the rotated position with angle θ. Applied force P remains always horizontal. The rigid bar is assumed massless. Ignore the gravity effect for simplicity. One may consider this system as a crude model of a launch vehicle on a test stand, with P and K representing thrust and torsional spring bending stiffness respectively.

- **(a)** Derive the equation of motion for θ.
- **(b)** Assume small θ and simplify the equation of motion to $I\ddot{\theta} + k_{eff}\theta = 0$ where k_{eff} is the effective spring constant. Show that $K_{eff} = K - PL$.
- **(c)** Show that the natural frequency of the system decreases as force P increases.

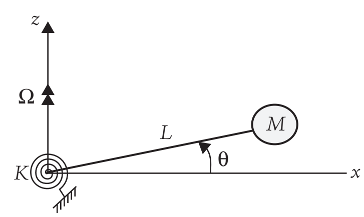


1.10 Consider ground vibration test of a small spacecraft weighing 5,300 N. For vertical vibration, the spacecraft is modeled as a SDOF system. The test results show that the period of damped free vibration is 1.2 *sec* and the amplitude of vertical displacement reduces to one-half in two cycles. Determine damping ratio ζ, damping constant c and spring constant k of the landing gear.

1.11 A seat with an occupant during crash landing is modeled as SDOF system undergoing vertical motion. The seat has a damper with adjustable damping. At a given damping ratio, the amplitude decays to 50% in one cycle. Determine the amplitude decay (in percentage) in one cycle if the damping ratio is doubled.

1.12 The sketch shows a mass M connected to a massless rigid bar of length L rotating about the z-axisat a constant speed of Ω rad/sec. The mass is also undergoing angular motion in the x-z plane. A rotational spring of constant K attached at the pivot point represents the bending stiffness of aslender body. The forces acting on the mass are the centrifugal force and the gravity force.

(a) Derive the equation of motion for θ.
(b) Assume small θ and simplify the equation of motion to $I\ddot{\theta}+k_{eff}\theta=0$ where k_{eff} is the effective spring constant.
(c) Show how the centrifugal force term contributes to the effective stiffness of the system.
(d) Does the natural frequency increases as the rotating speed increases?



1.13 A force is applied to a lightly damped SDOF system which was initially at rest. The recorded acceleration of ensuing motion is found to fit $\ddot{y}=A\cos\omega_n t$ where $A=1000\ m/\sec^2$ Neglect damping and determine the following:

(a) velocity and displacement as a function of time.
(b) applied force for mass $m=100\ kg$.

1.14 The elastic force on the landing gear after touchdown may exhibit a nonlinear stiffening behavior as a function of aircraft vertical displacement. In this case, the SDOF equation of motion for vertical displacement of the aircraft may be expressed as

$$m\ddot{y}+c\dot{y}+F_s(y)=mg$$

with y defined positive downward. $F_s(y)$ is a nonlinear spring force. Use a numerical integration technique such as the Euler method or the RK4 method to determine vertical displacement of an aircraft with the following information:

Aircraft weight: 2,100 lb.
Damping constant: 200 lb-s/in.
Sink velocity at touchdown: 15 ft/sec.
Spring force: $F_s(y)=Ky\left(1+0.25\hat{y}^2\right)$, where K = 725 lb/in and $\hat{y}=0$ if $|y|$ measured in inches is less than 3 inches; otherwise, $\hat{y}=|y|-3$.